The objects of study of this course are (quasi-projective) varieties over an algebraically closed field $k = \bar{k}$

Def A <u>variety over k</u> is a reduced, separated, finite type scheme over k.

A scheme X is <u>reduced</u> if $\mathcal{O}_X(U)$ is reduced for all $U \subseteq X$ open.

A scheme X is <u>integral</u> if $\mathcal{O}_X(U)$ is an integral domain for all $U \subset X$ open.

Equivalently, X is reduced and irreducible.

A morphism of schemes $f: X \to Y$ is

- <u>quasi-compact</u> if the preimage of every open subset of Y is quasicompact
- <u>locally of finite type</u> if for every open affine $\operatorname{Spec}(B) \subset Y$ and open affine $\operatorname{Spec}(A) \subset f^{-1}(\operatorname{Spec}(B))$ the induced morphism $B \to A$ is a finite type morphism of schemes
- <u>of finite type</u> if it is quasi-compact and locally of finite type.

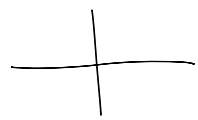
A <u>scheme over k</u> is a scheme X together with a morphism $X \to \operatorname{Spec}(k)$ (called structure morphism)

Morphisms of schemes over k are commutative diagrams

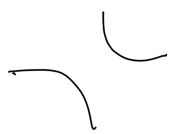
$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & \operatorname{Spec}(k) & \end{tikzcd}$$

Examples

$\mathrm{Spec}\left(k[x,y]/(xy)\right)$ is reducible, reduced, but not integral.



$\mathrm{Spec}\left(k[x,y]/(xy-1)\right)$ is integral.



Separatedness: "AG analogue of Hausdorff"

$f: X \to Y$ is separated if $\Delta: X \to X \times_{f,Y,f} X$ is a closed immersion.

$$\begin{tikzcd} X \times_{f,Y,f} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Fact: morphisms of affine schemes are separated

Cor.: Δ_f is always locally closed immersion.

$\Rightarrow$ $f: X \to Y$ is separated iff $\Delta(X) \subseteq X \times_Y X$ is closed.

"AG version of (relative) compactness"

Example Consider the line with two origins.

It is the scheme X obtained by gluing two copies of $\mathbb{A}^1$ along $\mathbb{A}^1 \setminus \{0\}$ by the identity:

$$X = U_1 \cup U_2 \qquad \text{where } U_i \cong \operatorname{Spec}(k[x])$$

and $\operatorname{Spec}(k[x^{\pm 1}]) \cong U_1 \cap U_2 \xrightarrow{\cong} U_2 \cap U_1 \cong \operatorname{Spec}(k[x^{\pm 1}])$

$$x \longmapsto x$$

X is not separated.

Properness $f: X \to Y$ is universally closed if for all $g: Z \to Y$

$f': X \times_Y Z \to Z$ is universally closed.

$f: X \to Y$ is proper if it is of finite type, separated, and universally closed.

Valuative criteria

A valuation on a field K is a map

$\nu: K^\times \to \Gamma$ to a totally ordered abelian group Γ

such that for all $a, b \in K^\times$

$$\nu(a+b) \geq \min(\nu(a), \nu(b))$$
$$\nu(ab) = \nu(a) + \nu(b)$$
$$\nu(1) = 0$$

A field K with a valuation is called a valued field.

(K, ν) valued field

Ring of integers: $R = \{a \in K^\times \mid \nu(a) \geq 0\} \cup \{0\}$

Example $k(t) \longrightarrow \mathbb{Z}$

$f = t^n g \longmapsto n$

> this idea is basically the same for any DVR w/ uniformizer t.

w/ valued ring $k[t]$

or $k((t)) \longrightarrow \mathbb{Z}$

$f = \sum a_n t^n \longmapsto \inf\{n \mid a_n \neq 0\}$

Valuative problem:

for a morphism $f: X \to Y$

(K, v) valued field w/ valued ring

Valuative test diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(R) \arrow[r] \arrow[ur, dashed, "\varphi"] & Y \end{tikzcd}$$

f
- satisfies the uniqueness part of the valuative criterion if there is at most one φ making the diagram commute.
- satisfies the existence part of the valuative criterion if there exists a φ making the diagram comm.

Theorem (Valuative Criterion)

Let $f: X \to Y$ is
(1) separated iff it satisfies the uniqueness v.c
(2) universally closed iff it satisfies the existence v.c.
(3) proper iff it satisfies both parts of the valuative criterion

Example $\mathbb{P}^1$ is universally closed.

$\mathrm{Spec}(K) \xrightarrow{\pi} U_1 \cap U_2 = \mathrm{Spec}(k[x^{\pm 1}])$

$\pi^b(x) \in R$ ✓

$\pi^b(x) < 0 \Rightarrow \pi^b(x^{-1}) \in R$ ✓

Challenge: Show $\mathbb{P}^1$ is separated. $\rightsquigarrow$ $\mathrm{Spec}(R) \to \mathrm{Spec}(k[x^{-1}])$

Dimension and regularity.

X irreducible scheme $/k$

$$\dim(X) := \sup\{\dim(\mathcal{O}_{X,x}) \mid x \in X\}$$

$Y \subseteq X$ irreducible closed subscheme

$$\mathrm{codim}_X(Y) := \dim(\mathcal{O}_{X,Y})$$

Recall bijection $X \longleftrightarrow \{Y \subseteq X \text{ irreducible}\}$

A scheme X is regular at $x \in X$ if

$$\dim(\mathcal{O}_{X,x}) = \dim_{k(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2)$$

a scheme is regular if it is regular at all points.

- connected + regular $\Rightarrow$ irreducible
- regular $\Rightarrow$ reduced, normal.

Theorem (Jacobian criterion)

Suppose $U = \mathrm{Spec}(k[x_1, \dots, x_n]/(f_1, \dots, f_m))$ has dimension d. Then U is regular @ p iff

$$\left(\frac{df_j}{dx_i}\right)_{i,j} \text{ has corank } d \text{ @ } p.$$